EXAM SOLUTIONS

Q6.

The general clockwise rotation matrix in $\mathbb{R}^2$ is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

We have $\theta = \frac{3\pi}{2}$, and

$$\cos\left(\frac{3\pi}{2}\right) = 0, \sin\left(\frac{3\pi}{2}\right) = -1$$

So our particular rotation matrix is

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Clearly, the linear transformation that reflects a vector across the vertical axis changes the first standard basis vector, $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, to $\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$.

This corresponds to the linear transformation (matrix)

$$S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix composition, $\circ$, is an equivalent notion to matrix multiplication. Therefore, we have the two compositions.

1. $T \circ S$, the linear transformation corresponding to a reflection followed by rotation:

$$\begin{aligned} T \circ S &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\text{using column by coordinate rule} \\ &= \left(\left(-1\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) + 0\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)\right) \ \left(0\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) + 1\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)\right)\right) \\ &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

2. $S \circ T$, the linear transformation corresponding to rotation, followed by reflection. Since matrix composition is generally not commutative, we obtain a different matrix.

$$\begin{aligned} S \circ T &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &\text{using column by coordinate rule} \\ &= \left(\left(0\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) + 1\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right) \ \left(-1\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right) + 0\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)\right)\right) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Q7.

7.1 The matrix A corresponds to a linear transformation $T : \mathbb{R}^4 \mapsto \mathbb{R}^3$. A has 3 rows and 4 columns, so its matrix-vector multiplication is only defined when with vectors in $\mathbb{R}^4$. Accordingly, it will output a vector in $\mathbb{R}^3$.
So, $p = 4$.

7.2 See above explanation. $q = 3$.

7.3 To find all vectors $\vec{x} \in \mathbb{R}^4$ whose image under T is $\vec{b}$, we seek all solutions $\vec{x} = (x_1, x_2, x_3, x_4, x_5)^T$ to the equation

$$T\vec{x} = \vec{b}$$

We can do this using our usual row reduction methods.

$$\left(\begin{array}{cccc|c} -2 & 3 & 7 & -11 & -3 \\ 1 & 0 & -2 & 1 & 3 \\ 1 & -1 & -3 & 4 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -3 & 4 & 2 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We now have the augmented matrix in echelon form. So, x_4 and x_3 are free. Then, let $s, t \in \mathbb{R}$ be free variables

$$\begin{aligned} x_1 &= 3 - t + 2s \\ x_2 &= 1 + 3t - s \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

So, all vectors $\vec{x}$ are of the form

$$\vec{x} = \begin{pmatrix} 3 + 2s - t \\ 1 - s + 3t \\ s \\ t \end{pmatrix}$$